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SHORTER COMMUNICATIONS

A SYNTHESIS OF ANALYTICAL RESULTS FOR NATURAL CONVECTION HEAT TRANSFER ACROSS RECTANGULAR ENCLOSURES

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NOMENCLATURE

A,	aspect ratio, H/L;
g,	gravitational acceleration;
H,	vertical dimension;
k,	thermal conductivity;
K,	permeability of porous medium;
L,	horizontal dimension;
Nu,	Nusselt number;
Pr,	Prandtl number;
Q,	heat transfer rate;
Ra_L,	Rayleigh number based on L;
$Ra_{L,K}$,	Rayleigh number based on L and K;
T_a,	temperature of warm wall;
T_b,	temperature of cold wall;
α,	thermal diffusivity;
β,	coefficient of thermal expansion;
ν,	kinematic viscosity.

1. INTRODUCTION

NATURAL convection is an important heat-transfer mechanism in the technology of building insulation. From the point of basic research in heat transfer, the phenomenon is being studied mainly in terms of simple models of free convection in rectangular enclosures filled either with a Newtonian fluid or with a fluid-saturated porous medium. The subject of free convection in enclosures is extensive and has numerous applications in practical engineering situations. A comprehensive review of free convection heat transfer in enclosures filled with fluid was presented in a monograph by Ostrach [1]. In a more recent article, Buchberg, Catton and Edwards [2] reviewed the applications pertaining to heat transfer through spaces encountered in the solar power technology.

The purpose of this article is to review the analytical work on free convection in rectangular enclosures. The review consists of illustrating the ability of each theory to predict the dependence of the net (wall-to-wall) heat-transfer rate on the cavity aspect ratio height/length (H/L). As most analytical studies have been conducted in either the vertical or horizontal cavity limit, each theory has not made it clear how the variation in aspect ratio influences the net heat-transfer rate through the vertical double wall. This article presents an overall view of what has been accomplished analytically over the entire geometry spectrum (tall cavities, $H \gg L$, and shallow cavities, $H \ll L$). In addition to immediate comparisons among different theories, this review points out which portions (bands) of the geometry spectrum need more analytical work. An essential aspect of this review is the parallel coverage of enclosures filled with fluid (Fig. 1), *vis-à-vis* enclosures filled with a porous medium (Fig. 2). In this regard, the symmetry between Figs. 1 and 2 is significant, meaning that analytical methods developed in one field can

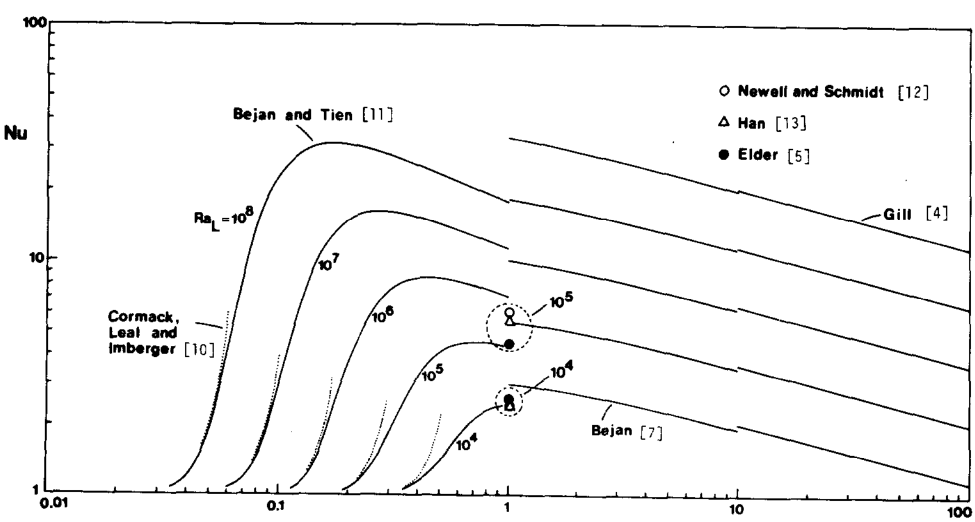

Fig. 1. Summary of heat-transfer theories for a rectangular cavity filled with Boussinesq-incompressible fluid.

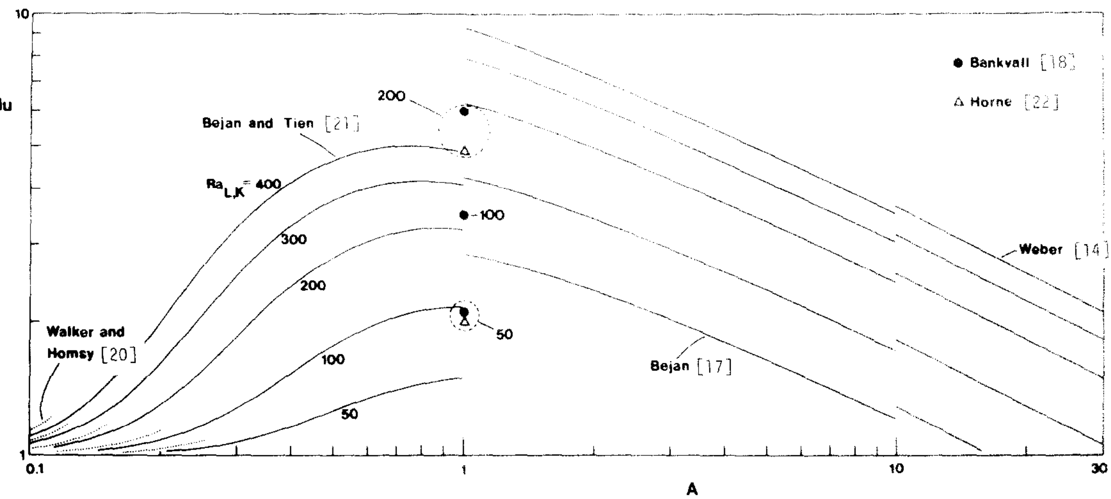

FIG. 2. Summary of heat-transfer theories for a cavity filled with a fluid-saturated porous medium.

be successfully adopted in the treatment of the related problem in the parallel field.

2. RECTANGULAR ENCLOSURES FILLED WITH FLUID

Bachelor [3] showed that the fluid mechanics of the fluid-filled rectangular cavity depends on three dimensionless parameters,

$$Ra_L = \frac{g\beta L^3(T_a - T_b)}{\alpha\nu}, \quad Pr = \nu/\alpha, \quad A = H/L. \qquad (1,2,3)$$

In the above definitions, $A, g, H, L, Pr, Ra_L, T_{a,b}, \alpha, \beta, \nu$ are the aspect ratio, gravitational acceleration, cavity vertical dimension, horizontal dimension, Prandtl number, Rayleigh number, vertical wall temperatures, thermal diffusivity, coefficient of thermal expansion and kinematic viscosity, respectively. The Nusselt number for net heat transfer Q between the two vertical walls of the enclosure is defined as

$$Nu = \frac{Q}{kH(T_a - T_b)/L} \qquad (4)$$

where k is the thermal conductivity of the medium. In general we expect Nu to be a function of Ra_L, Pr and A. However, for fluids with Prandtl number of order one and higher, experimental results indicate that the Pr effect of Nu is minor so that as a good approximation Nu depends only on Ra_L and A.

2(a) *Tall enclosures*

The only theory available for predicting the Nusselt number for tall cavities in the range where free convection is the dominant heat transfer mechanism was proposed by Gill [4]. He envisioned a boundary layer-type flow regime in which the fluid motion is confined to layers near the two vertical surfaces, leaving the fluid in the central portion of the vertical space in a stagnant and vertically stratified condition. It must be said that flow visualization experiments starting with the work of Elder [5] and numerical simulations of the flow such as those reported by de Vahl Davis [6] provide little support for the existence of a truly stagnant and stratified core as assumed by Gill. However, a recent comparison [7] of Gill's theoretical prediction for Nu with experimental and numerical correlations shows excellent agreement in the Ra_L range where convection heat transfer is dominant.

The overall Nusselt number predicted by Gill's theory, as calculated later by Bejan [7], is

$$Nu = 0.364(Ra_L/A)^{1/4}. \qquad (5)$$

Figure 1 shows this prediction for a number of Ra_L values and aspect ratios in the range $10 < A < 100$. As the vertical cavity becomes shorter (i.e. as A decreases), the heat-transfer rate at constant wall thickness L and temperature difference (constant Ra_L) increases. An explanation for why the heat-transfer rate increases as the convective cell is shortened can be given in terms of the counterflow heat exchanger formed by the two vertical branches of the convective cell. With one (warm) branch rising and the other (cold) falling, the counterflow transports heat vertically upward between the two walls. The heat flow is first extracted from the lower left-hand corner of the cavity, is then carried upward and, eventually deposited in the upper right-hand corner. As the end-to-end insulation capability of any counterflow heat exchanger increases with the heat-transfer area available between the two branches [8], it is clear that the longer the flow path the lower the convective heat leak carried by the counterflow, hence the lower the net heat transfer across the vertical enclosure.

An extension of Gill's analysis was proposed more recently by Bejan [7] who, like Quon [9], questioned the basis for selecting the two floating constants which appear in the Gill theory. Since Gill's solution cannot satisfy at the same time the adiabatic and impermeable conditions along the two horizontal boundaries of the cavity, Bejan proposed to account for all four conditions in an average manner by requiring that the net upflow of energy described in the preceding paragraph vanishes in the vicinity of the top and bottom end walls. The impact of this modification is improved agreement between Gill's theory and experimental results for cases in which the cavity is moderately tall.

2(b) *Shallow enclosures*

The case of horizontal cavities with adiabatic horizontal walls and isothermal (T_a, T_b) vertical end-walls was considered theoretically by Cormack *et al.* [10]. Their analysis was based on an asymptotic solution in parameter $A \ll 1$ for the flow and temperature field in the core of the cavity, matched with asymptotic solutions describing the flow in the two end-turn regions. The Nusselt number resulting from this theory,

$$Nu = 1 + 2.86 \times 10^{-6}\, Ra_L^2 A^8, \qquad (6)$$

was found to agree well with experimental and numerical results in the limit $A \to 0$ and Ra_L finite but fixed. The dotted lines of Fig. 1 show the asymptotic result (6). The curves demonstrate that decreasing the cavity height to the point where the cavity is very shallow has the effect of quenching the convective cell and the heat leak associated with it. This conclusion is consistent with the counterflow heat exchanger analogy presented in the preceding subsection, the only difference being that this time the counterflow pattern is oriented horizontally.

Bejan and Tien [11] constructed a theory for the same phenomenon by coupling the asymptotic ($A \to 0$) core solution of Cormack *et al.* with integral solutions for the flow in the two end regions. The result of this matching is a solution whose range of validity extends well into the aspect ratio domain where the asymptotic solution [10] breaks down. The Nusselt number predicted by this integral-asymptotic analysis agrees very well with all experimental and numerical information available on the phenomenon, especially in the high Ra_L range where the Cormack *et al.* asymptotic solution does not apply. The solution is presented on the same Nu–A graph of Fig. 1. This is the first instance where it is shown analytically that at constant Ra_L, there exists a well defined aspect ratio for which the overall heat transfer rate (Nu) is a maximum.

2(c) *Square enclosures*

No theory exists for free convection in enclosures of aspect ratio equal to one. However, we consider here the case $H/L = 1$ in order to show that the theories for tall enclosures and shallow enclosures approach each other in this limit. This fact is evident from Fig. 1 where, in addition, we show heat-transfer results based on the experimental and numerical work published by Newell and Schmidt [12], Han [13] and Elder [5]. The tall enclosure Nusselt number prediction approaches the experimental results from above while the shallow enclosure theory approaches the same results from below.

3. RECTANGULAR ENCLOSURES FILLED WITH FLUID-SATURATED POROUS MEDIUM

The theoretical study of free convection in rectangular enclosures packed with porous material has traditionally lagged behind the theoretical advances reviewed in the preceding section. The basic assumption in the analytical treatment of the phenomenon is that the Darcy flow model applies. Coupled with the Boussinesq-incompressible fluid model, the Darcy flow assumption leads to a set of linear momentum equations. However, the mathematical problem remains weakly nonlinear due to the convective heat transport terms present in the energy equation. The free convection mechanism depends on two dimensionless groups, the aspect ratio A and the Darcy-modified Rayleigh number:

$$Ra_{L,K} = \frac{g\beta KL(T_a - T_b)}{\alpha\nu} \qquad (7)$$

where K is the permeability of the medium.

3(a) *Tall enclosures*

The equivalent of Gill's theory [4] for convection in a vertical porous slot was reported a few years later by Weber [14] whose theoretical result for the Nusselt number,

$$Nu = 3^{-1/2}(Ra_{L,K}/A)^{1/2}, \qquad (8)$$

agrees fairly well with the experimental data reported by Schneider [15] and Klarsfeld [16], particularly in cases where A is large and Nu is clearly dominated by convective effects ($Nu \gg 1$).

In a recent note, Bejan [17] modified the Weber theory, fitting the boundary layer solution with average zero energy flux conditions along the top and bottom walls. The Nusselt number predicted by the modified theory [17] agrees extremely well with the experimental data of [15, 16] as well as with numerical heat transfer calculations reported in [18, 19].

Shallow cavities

Walker and Homsy [20] have recently published an asymptotic analysis in the limit $A \to 0$ and $Ra_{L,K}$ finite, patterned after the earlier technique developed by Cormack *et al.* For the Nusselt number, Walker and Homsy found

$$Nu = 1 + \frac{1}{120} Ra_{L,K}^2 A^4. \qquad (9)$$

The same result was published in an independent study by Bejan and Tien [21]. The dotted lines on Fig. 2 show the asymptotic result (9) and its limited domain of applicability. In their study, Bejan and Tien [21] presented also an approximate solution which consists of matching the $A \to 0$ asymptotic solution valid in the core of the shallow cavity with Karman–Pohlhausen solutions valid in the end-turn regions. This result is presented in Fig. 2, again, demonstrating analytically the existence of a critical aspect ratio for which, at constant $Ra_{L,K}$, the Nusselt number reaches a maximum.

3(c) *Square enclosures*

The case $H/L = 1$ is considered here to show how the tall and shallow enclosure theories approach one another when the enclosure is neither tall nor shallow. The discrepancy between the two theoretical predictions is relatively more pronounced than in Fig. 1. However, the two predictions fall on either side of the numerical heat-transfer data reported for square cavities by Bankvall [18] and Horne [22].

4. CONCLUDING REMARKS AND RECOMMENDATIONS FOR FUTURE RESEARCH

The analytical work reviewed above leads to the practical conclusion that sufficient analytical means are available for predicting heat-transfer rates and the effect of cavity shape on heat transfer. The accuracy of these theoretical results increases in the two extreme cases $A \gg 1$ and $A \ll 1$. For a given temperature difference and vertical wall spacing, constant Ra_L or $Ra_{L,K}$, there exists a band of aspect ratios which lead to the highest possible heat-transfer rates through the cavity. From a thermal insulation engineering point of view this range of aspect ratios must be avoided. In order to reduce the convective contribution to heat transfer we must force the convective cell to function as an elongated counterflow heat exchanger in which the two long branches are in superior mutual thermal contact.

Examining Figs. 1 and 2 we recognize areas in which further theoretical work would constitute a contribution. First, the limit $Ra \to 0$ with $A \gg 1$ and fixed lacks a theory describing how the heat-transfer rate shifts from the conduction dominated regime to the convection (boundary layer) regime analyzed by Gill and Weber. An appropriate start in this direction would be the asymptotic analyses of Batchelor [3] and, for porous cavities, Burns, Chow and Tien [19], who discuss qualitatively the variation of Nu with Ra_L and $Ra_{L,K}$ in the conduction dominated limit. Another area ready for theoretical development is the square cavity problem.

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NATURAL CONVECTION IN A VERTICAL CYLINDRICAL WELL FILLED WITH POROUS MEDIUM

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NOMENCLATURE

$a_0, a_2, a_4,$ $a_6, b_0,$	coefficients;
c_p,	specific heat at constant pressure;
g,	gravitational acceleration;
h,	thermal conductivity;
k,	permeability;
l,	length of similarity pattern;
L,	depth of well;
Nu,	Nusselt number;
Q,	heat-transfer rate through well opening;
r,	radial position;
R,	well radius;
Ra_L,	Rayleigh number;
T,	temperature;
T_1,	well-wall temperature;
T_2,	reservoir temperature;
$\hat{T}$,	temperature, Oseen solution;
u,	vertical velocity;
$\hat{u}$,	vertical velocity, Oseen solution;
$\hat{u}_\infty$,	vertical velocity infinitely far from wall;
v,	radial velocity;
w,	velocity perpendicular to the wall;
x,	vertical position;
y,	distance from vertical wall;
Y,	total thickness of vertical boundary layer;
$(\ )^*$,	dimensional quality;
$(\ )_c$,	pertaining to the core.

Greek symbols

α,	thermal diffusivity;
β,	coefficient of thermal expansion;
γ,	core radius;
δ,	horizontal length scale, Oseen solution;
μ,	viscosity;
ν,	kinematic viscosity;
ρ,	density;
ψ,	stream function.

1. INTRODUCTION

BOUYANCY-INDUCED convection in fluid-saturated porous media is an important topic in contemporary heat-transfer research. The objective of this article is to outline an analysis of the natural convection mechanism in a vertical cylindrical well filled with porous medium [1]. The well opens into a semi-infinite space filled with the same porous medium. In what follows we analyze the convection pattern generated when the cylindrical wall and the semi-infinite space are maintained at different temperatures.

The present problem is related to the work of Minkowycz and Cheng on convection about a vertical cylinder [2] and about a vertical plane [3]. The Minkówycz and Cheng studies, as well as the present one, are aimed at explaining the interaction between a very large porous reservoir and an irregular impermeable surface bordering the reservoir from above or below. The impermeable surface may protrude into the porous medium, as in [2,3], or it may have concavities filled by the neighboring porous material. The latter set of circumstances is the subject of the present investigation.

2. MATHEMATICAL FORMULATION

We model the fluid-saturated porous medium as homogeneous [4] with the following physical properties: fluid density, ρ; viscosity, μ; coefficient of thermal expansion, β;